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The far field of an oscillating airfoil in supersonic flow

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An approach utilizing multiple scales and matched asymptotic expansions is developed for the description of small perturbations at large distances from a thin airfoil oscillating harmonically in a uniform supersonic flow. The problem of determining the unsteady perturbation potential is formulated in general, and an analytical solution is derived for an airfoil with parabolic or flat surfaces. The results describe the flow ahead of the region influenced by the trailing edge. The variation in the pressure jump across an attached leading-edge shock wave is also obtained.

1. Introduction

For steady supersonic flow past a thin airfoil, it is well known (Lighthill 1954; Van Dyke 1975) that the displacement of characteristics and shock waves from their linearized positions is no longer small in comparison with the chord length if the distance from the airfoil is sufficiently large. If now the airfoil undergoes simple harmonic oscillations of small amplitude, linear acoustics provides a correct first approximation (Garrick & Rubinow 1946; Van Dyke 1954) for the flow perturbations at distances of the same order as the chord length. At large distances, however, the gradual distortion of the steady flow also influences the unsteady disturbances. Kurosaka (1977) studied these effects, with emphasis on high reduced frequencies and without explicit discussion of the decay of the unsteady pressure jump across a leading-edge shock wave. The purpose of this note is to complete the flow description in a systematic way for reduced frequencies of order one and to show how the pressure oscillations at a leading-edge shock wave will decrease in strength at large distances.

The problem formulation is given in §2 and the solution is carried out in §3. The method of solution is similar to the multiple-scales method used by Luke (1966) in a study of weakly nonlinear dispersive waves, also discussed by Kevorkian & Cole (1981). The present derivation differs in a number of details, as in the replacement of a periodicity requirement with a matching condition. An analytical solution is obtained for airfoils with parabolic or flat surfaces. The results describe the flow far from the airfoil and ahead of the region influenced by the trailing edge, except that a different solution is required close to a shock wave from the leading edge. A composite is easily formed, and remains correct near the shock wave all the way to the leading edge. The variation in the pressure jump along the shock wave is then considered in §4.

2. Formulation

Rectangular coordinates x and y are measured along and normal to the undisturbed flow direction respectively, with the origin at the steady-state position of the airfoil leading edge. The coordinates x and y, the time t, the velocity vector $\boldsymbol{q}$ and its magnitude q, the sound speed a, and the reduced frequency k are all non-dimensional, with the airfoil chord length and the undisturbed fluid velocity serving as the reference quantities. The pressure perturbation p is referred to twice the dynamic pressure in the undisturbed flow, γ is the ratio of specific heats, M is the undisturbed value of the Mach number, and $B^2 = M^2-1$.

For irrotational isentropic flow the velocity potential Φ satisfies

$$a^2 \operatorname{div} \boldsymbol{q} = \tfrac{1}{2}(\boldsymbol{q}\cdot\nabla q^2)+2\boldsymbol{q}\cdot\boldsymbol{q}_t+\Phi_{tt}, \tag{1}$$

where $\boldsymbol{q} = \nabla\Phi$ and

$$a^2 = \frac{1}{M^2}+\tfrac{1}{2}(\gamma-1)(1-q^2)-(\gamma-1)\,\Phi_t. \tag{2}$$

The pressure can be found from

$$1+\gamma M^2 p = (Ma)^{2\gamma/(\gamma-1)}. \tag{3}$$

A wedge-shaped leading edge will be assumed, so that the shock waves are attached. Since the shock waves are everywhere very weak, the largest terms in the solution can be obtained using (1)–(3). It is sufficient to consider the flow for $y > 0$. If the upper airfoil surface is defined by $S(x,y,t) = 0$, the boundary condition requiring zero flow through the surface is

$$S_t+\boldsymbol{q}\cdot\nabla S = 0, \tag{4}$$

at $S = 0$. For simple harmonic oscillations, the surface shape will be expressed in the form

$$S(x,y,t) = 0 = y-\epsilon f(x)-\delta\, \mathrm{e}^{\mathrm{i}kt}h(x), \tag{5}$$

for $0 < x < 1$, where $\delta \ll \epsilon \ll 1$, so that the amplitudes of the unsteady perturbations are small in comparison with the corresponding steady-state changes. Solutions will be sought for the portion of the flow ahead of the region influenced by the trailing edge.

The potential Φ and pressure p have various asymptotic representations, corresponding to different limits. In general, Φ and p can each be written as the sum of a steady-state part, a linearized time-dependent part, and terms of higher order:

$$\Phi = x+\epsilon\bar{\phi}(x,y;\epsilon)+\delta\, \mathrm{e}^{\mathrm{i}k(t-M^2x/B^2)}\tilde{\phi}(x,y;k,\epsilon)+O(\delta^2), \tag{6}$$

$$p = \epsilon\bar{p}(x,y;\epsilon)+\delta\, \mathrm{e}^{\mathrm{i}k(t-M^2x/B^2)}\tilde{p}(x,y;k,\epsilon)+O(\delta^2), \tag{7}$$

where $\bar{\phi}$, $\tilde{\phi}$, $\bar{p}$ and $\tilde{p}$ are bounded as $\epsilon\to 0$ and possess different forms of expansion as $\epsilon\to 0$ depending on the behaviour of x, y and k.

For x, y fixed as $\epsilon\to 0$, the first approximation for $\bar{\phi}$ is the steady-state linear-theory solution $\bar{\phi} = -f(x-By)/B$. But for $\epsilon y, x-By$ fixed as $\epsilon\to 0$ the cumulative effect of the small error in the slopes of the linearized characteristics can no longer be neglected. Rewriting the differential equation for $\bar{\phi}$ in terms of variables $\xi = x/B-y$ and y shows that in the first approximation a quantity X is constant along characteristics, and the velocity and pressure perturbations are again given by the linearized simple-wave solution but now with $\overline{\phi}_\xi$ = constant along the corrected characteristics:

$$B\xi = x - By = X + \frac{1}{2}\frac{(\gamma+1)\,M^4\epsilon y\overline{\phi}_\xi(X)}{B^2}, \tag{8}$$

$$p = -\frac{\overline{\phi}_\xi(X)}{B} = \frac{f'(X)}{B}. \tag{9}$$

For x, y, k fixed as $\epsilon \to 0$, the equations defining $\tilde{\phi}$ become

$$B^2\tilde{\phi}_{xx} - \tilde{\phi}_{yy} + \frac{k^2M^2}{B^2}\tilde{\phi} = O(\epsilon), \tag{10}$$

$$\tilde{\phi}_y(x,0) - \mathrm{e}^{\mathrm{i}kM^2x/B^2}V(x) = O(\epsilon), \tag{11}$$

where
$$V(x) = h'(x) + \mathrm{i}kh(x), \tag{12}$$

and the flow is undisturbed for $x < 0$. If the airfoil undergoes a rigid plunging and pitching oscillation, $h(x)$ is linear in x. The solution to the linearized problem for $\tilde{\phi}$, with the right-hand sides neglected in (10) and (11), is (Van Dyke 1954):

$$\tilde{\phi} = -\frac{1}{B}\int_0^{x-By} \mathrm{e}^{\mathrm{i}kM^2\sigma/B^2} J_0\left\{\frac{kM}{B^2}((x-\sigma)^2 - B^2y^2)^{\frac{1}{2}}\right\} V(\sigma)\,\mathrm{d}\sigma. \tag{13}$$

If $\xi y \to \infty$ with $\xi/y \to 0$, (13) becomes, after integration by parts and expansion of the resulting Bessel function,

$$\tilde{\phi} \sim -\frac{1}{\pi^{\frac{1}{2}}}\left(\frac{2B}{kM}\right)^{\frac{3}{2}} V(0)\frac{\xi^{\frac{1}{4}}}{(2y)^{\frac{3}{4}}}\cos\left(\frac{kM}{B}(2\xi y)^{\frac{1}{2}} - \frac{3\pi}{4}\right). \tag{14}$$

The first approximation is proportional to $V(0)$, and so represents waves originating from the leading edge. The asymptotic far-field solution must match with (14) and so also will contain only waves coming from the leading edge.

It is convenient now to transform from coordinates x and y to coordinates ξ and y. For $y \to \infty$ and $\xi/y \to 0$, retaining the largest terms of the right-hand side of (10) is found to give

$$\tilde{\phi}_{\xi y} + \frac{k^2M^2}{2B^2}\tilde{\phi} = \frac{(\gamma+1)\,\epsilon M^4}{2B^3}\left\{-(\overline{\phi}_\xi\tilde{\phi}_\xi)_\xi + \frac{2\mathrm{i}k}{B}\overline{\phi}_\xi\tilde{\phi}_\xi\right\} + \dots, \tag{15}$$

where derivatives with respect to y are of higher order and have been neglected. Terms proportional to $\epsilon\tilde{\phi}$ would lead to a change in the wavenumber that remains of higher order when $y = O(1/\epsilon)$, and so also have been omitted. The characteristics of (15) are the lines X = constant, with slopes affected by the $\overline{\phi}_\xi\tilde{\phi}_{\xi\xi}$ term in the same manner as for steady flow, implying again a cumulative effect that cannot be neglected when y is large. This suggests a further transformation, from ξ and y to X and y. Equation (15) now becomes

$$\tilde{\phi}_{Xy} + \frac{k^2M^2}{2B^2}\xi_X\tilde{\phi} = \epsilon\frac{(\gamma+1)\,M^4}{B^4}\mathrm{i}k\overline{\phi}_\xi\tilde{\phi}_X + \dots. \tag{16}$$

This equation is to be solved for $X = O(1)$ and $y = O(1/\epsilon)$, with the requirement that the solution be consistent with (14) when $1 \ll y \ll 1/\epsilon$.

3. Solution

The form of the solution to (16) for large y is suggested by the form of (8) and (14). If $x = O(1)$ and $y = O(1/\epsilon)$, the second term in the definition (8) of X is $O(1)$ and the expansion (14) of the linearized solution varies rapidly because the argument of the cosine is $O(\epsilon^{-\frac{1}{2}})$. Thus it is anticipated that $\tilde{\phi}$ can be written in terms of X and a 'slow' variable $\eta = \epsilon y$ as well as a 'fast' variable $\theta = \epsilon^{-\frac{1}{2}}\Theta(X, \eta)$, where Θ is to be determined. This formulation in terms of multiple scales is similar to that given by Luke (1966) for an example illustrating propagation of nonlinear dispersive waves. In accordance with this procedure X, η and θ are to be treated as independent variables and the differential equation (16) for $\tilde{\phi}$ becomes

$$\frac{k^2M^2}{2B^2G^2}\xi_X(\tilde{\phi}_{\theta\theta}+G^2\tilde{\phi})$$
$$= -\epsilon^{\frac{1}{2}}\left(\Theta_\eta\tilde{\phi}_{X\theta}+\Theta_X\tilde{\phi}_{\eta\theta}+\Theta_{X\eta}\tilde{\phi}_\theta-\frac{(\gamma+1)M^4}{B^4}ik\bar{\phi}_\xi\Theta_X\tilde{\phi}_\theta\right)+\ldots, \quad (17)$$

where
$$\eta = \epsilon y, \quad \theta = \epsilon^{-\frac{1}{2}}\Theta(X,\eta), \quad (18)$$

and G is defined by
$$\Theta_X\Theta_\eta G^2(X,\eta) = \frac{1}{2}\frac{k^2M^2\xi_X}{B^2}, \quad (19)$$

with ξ_X found as a function of X and η from (8).

An expansion of $\tilde{\phi}$ is now assumed in the form
$$\tilde{\phi} = \epsilon^{\frac{3}{4}}\tilde{\phi}_1(X,\eta,\theta)+\epsilon^{\frac{5}{4}}\tilde{\phi}_2(X,\eta,\theta)+\ldots. \quad (20)$$

The choice $\tilde{\phi} = O(\epsilon^{\frac{3}{4}})$ is made so that the difference between $\tilde{\phi}$ and the solution (14) can be made arbitrarily small if ϵ, $1/y$ and η are sufficiently small. Since $\tilde{\phi}$ and (14) do not possess limits separately, this 'matching' represents a slight generalization of the matching of limit-process expansions. The second term in $\tilde{\phi}$ must be $O(\epsilon^{\frac{5}{4}})$ because the terms on the right-hand side of (17) are of this order.

Terms $O(\epsilon^{\frac{3}{4}})$ and $O(\epsilon^{\frac{5}{4}})$ in (17) give, for $j = 1$ and $j = 2$ respectively,
$$\tilde{\phi}_{j\theta\theta}+G^2\tilde{\phi}_j = d_j, \quad (21)$$

where $d_1 = 0$ and d_2 depends on $\tilde{\phi}_1$. Since the differentiation with respect to θ is carried out with X and η held fixed, G is treated as a constant for the integration of (21). The solution for $\tilde{\phi}_1$ can be written as
$$\tilde{\phi}_1 = A(X,\eta)\cos\{G(X,\eta)\theta-\chi(X,\eta)\}, \quad (22)$$

and the functions G, χ and A are to be determined. It is then found that d_2 contains terms proportional to $\cos(G\theta-\chi)$, $\sin(G\theta-\chi)$ and $\theta\sin(G\theta-\chi)$. If no secular terms are to be present in $\tilde{\phi}_2$, i.e. if $\tilde{\phi}_2$ is to remain finite as $\theta\to\infty$, the coefficients of these terms must all be zero. These three conditions provide differential equations for G, χ and A:
$$\Theta_X G_\eta+\Theta_\eta G_X = 0, \quad \Theta_X\chi_\eta+\Theta_\eta\chi_X = 0, \quad (23)$$

$$\Theta_X A_\eta+\Theta_\eta A_X+\left\{\Theta_{X\eta}+\frac{(\gamma+1)ikM^4f'(X)\Theta_X}{B^4}\right\}A = 0. \quad (24)$$

From (23), G and χ are constant along characteristics $dX/d\eta = \Theta_\eta/\Theta_X$. As $\eta\to 0$, $G\theta-\chi$ must match with the argument of the cosine in the solution (14) for large ξy;

it follows that $\chi \sim \frac{3}{4}\pi$ and $G\Theta \sim (kM/B^2)(2BX\eta)^{\frac{1}{2}}$ as $\eta \to 0$. Combining this expression for $G\Theta$ with the definition (19) of G and substituting $G^2\Theta_X\Theta_\eta \sim$ constant as $\eta \to 0$ in the differential equation for G leads to

$$4\frac{X\Theta_X}{\Theta}\frac{\eta\Theta_\eta}{\Theta} \sim 1 \sim \frac{X\Theta_X}{\Theta}+\frac{\eta\Theta_\eta}{\Theta}. \tag{25}$$

Solving then yields $\Theta \sim (\text{const.})(X\eta)^{\frac{1}{2}}$ as $\eta \to 0$ and so $G \sim$ constant. In (19) a constant factor in G was left arbitrary (if G is multiplied by C, Θ is divided by C), and so the constant value of G as $\eta \to 0$ is arbitrary. We will choose $G \sim kM/B^2$, so that

$$\Theta \sim (2BX\eta)^{\frac{1}{2}}, \tag{26}$$

as $\eta \to 0$. The slopes of the characteristics become $\mathrm{d}X/\mathrm{d}\eta \sim X/\eta$, and so as $\eta \to 0$ the characteristics are straight lines through the origin in the (X, η)-plane. The constant values of G and χ provide initial conditions for (23), and we can conclude that G and χ are constant everywhere. Equation (19) now becomes

$$2\Theta_X\Theta_\eta = B-\frac{(\gamma+1)M^4\eta f''(X)}{2B}. \tag{27}$$

This differential equation for the phase function Θ is analogous to the dispersion relation in Luke's (1966) example, which was obtained with the help of a periodicity requirement. Here, however, it did not appear possible to conclude that $G =$ constant directly.

An analytical solution to (27) can be obtained for airfoil shapes having the form

$$f(x) = a_1x+a_2x^2, \tag{28}$$

for $0 < x < 1$ and $a_1 > 0$. For a parabolic airfoil at zero incidence $a_2 = -a_1$; for a wedge $a_2 = 0$. Since $a_1 > 0$ the region of disturbed flow is bounded by a weak shock wave from the leading edge. If f has the quadratic form (28), $\overline{\phi}_\xi(X) = -(a_1+2a_2X)$ and ξ_X is a function only of η. Then (27) can be rewritten

$$2\Theta_X\Theta_Y = 1, \tag{29}$$

where $Y_\eta = B^2\xi_X$ and

$$Y = B\eta-\frac{(\gamma+1)M^4a_2\eta^2}{2B}. \tag{30}$$

Equation (29) has the form $g(X, Y, \Theta, \Theta_X, \Theta_Y) = 0$, with $\partial g/\partial X = \partial g/\partial Y = \partial g/\partial\Theta = 0$. If s is measured along characteristics, one then obtains

$$0 = \frac{\mathrm{d}\Theta_X}{\mathrm{d}s} = \frac{\mathrm{d}\Theta_Y}{\mathrm{d}s} = \frac{\mathrm{d}X}{\mathrm{d}s}-2\Theta_Y = \frac{\mathrm{d}Y}{\mathrm{d}s}-2\Theta_X = \frac{\mathrm{d}\Theta}{\mathrm{d}s}-4\Theta_X\Theta_Y. \tag{31}$$

From the first two parts of (31), Θ_X and Θ_Y are seen to be constant along characteristics. From the next two parts, and with the requirement of matching with (26), it is also found that Y/X is constant along characteristics. Finally, combining with the last part of (31) shows the solution to be

$$\Theta = (2XY)^{\frac{1}{2}}. \tag{32}$$

For more general airfoil shapes, a procedure similar to that indicated by (31) can be used to replace (27) with a system of ordinary differential equations to be solved numerically.

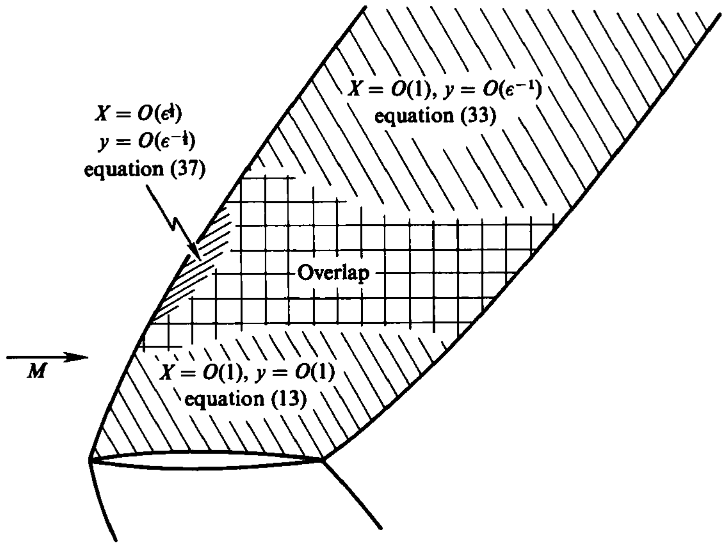

FIGURE 1. Domains of validity for different asymptotic representations.

With the solution (32), the result (22) for $\tilde{\phi}_1$ becomes

$$\tilde{\phi}_1 = A(X, Y) \cos\left\{\frac{\epsilon^{-\frac{1}{2}} kM(2XY)^{\frac{1}{2}}}{B^2} - \tfrac{3}{4}\pi\right\}, \tag{33}$$

where A now has been written as a function of X and Y. The differential equation (24) for A is

$$0 = XA_X + YA_Y + \left\{\frac{1}{2} + \frac{(\gamma+1)\,\mathrm{i}kM^4 Y(a_1+a_2 X)/B^4}{[B^2-2(\gamma+1)M^4 a_2 Y/B]^{\frac{1}{2}}}\right\} A. \tag{34}$$

Integration along the characteristics $\mathrm{d}Y/\mathrm{d}X = Y/X$ gives the solution

$$A = -\frac{B^2}{\pi^{\frac{1}{2}}}\left(\frac{2}{kM}\right)^{\frac{3}{2}} V(0)\frac{X^{\frac{1}{4}}}{(2Y)^{\frac{3}{4}}} \exp(-\mathrm{i}k\zeta), \tag{35}$$

where $$\zeta(X, Y) = \frac{M^4}{B^4}(\gamma+1)\,\eta\left\{a_1 + a_2 X\frac{B\eta}{Y}\left[1 - \frac{(\gamma+1)M^4 a_2 \eta}{3B^2}\right]\right\}, \tag{36}$$

with η and Y related by (30). The multiplicative integration constant has been chosen for agreement with (14).

The solution for the unsteady perturbation potential $\tilde{\phi}$ is then given by (33), with definitions of the variables obtained from (6), (8), (9), (18), (20), (28) and (30). The cosine factor in (33) describes variations having large wavenumber, $O(\epsilon^{-\frac{1}{2}})$ when X and Y are $O(1)$, with constant phase along lines $XY =$ constant and constant wavenumber along $Y/X =$ constant. The factor A describes 'slow' changes in amplitude and phase; the amplitude is constant along $Y^3/X =$ constant and is proportional to $Y^{-\frac{3}{4}}$ along the characteristics $X =$ constant of the steady flow.

An alternate form of solution is required in a region close to the shock wave from the leading edge for which $y = O(\epsilon^{-\frac{1}{2}})$ and $\xi = O(\epsilon^{\frac{1}{2}})$. Here the phase function Θ is of order one (i.e. XY is no longer large), and the displacement of characteristics from their linearized positions is again of the same order as the distance from the shock wave. The solution satisfies (16) without the right-hand side, rewritten in terms of coordinates $\epsilon^{-\frac{1}{2}}X$ and $\epsilon^{\frac{1}{2}}y$, and matches as $\epsilon^{\frac{1}{2}}y \to 0$ with the first term in the expansion of (13) for $y \to \infty$ with ξy held fixed. The result is

$$\tilde{\phi} \sim -\frac{1}{kM} V(0) \left(\frac{2BX}{y}\right)^{\frac{1}{2}} J_1 \left\{\frac{kM}{B^2}(2BXy)^{\frac{1}{2}}\right\}. \tag{37}$$

Domains of validity of the solutions (13), (33) and (37) are shown schematically in figure 1.

A composite formed from (33) and (37) describes the flow ahead of the region influenced by the trailing edge, for $XY = O(1)$ as well as $XY \gg 1$, provided always that $X \ll Y$:

$$\tilde{\phi} \sim -\frac{B}{kM} V(0) \left(\frac{2\epsilon X}{Y}\right)^{\frac{1}{2}} \mathrm{e}^{-\mathrm{i}k\zeta} J_1 \left\{\epsilon^{-\frac{1}{2}}\frac{kM}{B^2}(2XY)^{\frac{1}{2}}\right\}. \tag{38}$$

The results (33) and (38) include terms which are not present in Kurosaka's (1977) solution, since he omitted terms in the potential equation which did not involve k; his solution contains other terms which are not required in a first approximation when $X, Y, k = O(1)$. The time-dependent term in the pressure can now be found from (7), with $\tilde{p} \sim -\tilde{\phi}_x$. Substituting (38) leads to

$$\tilde{p} \sim (B^2\xi_X)^{-1} \mathrm{e}^{-\mathrm{i}k\zeta} V(0) J_0 \left\{\epsilon^{-\frac{1}{2}}\frac{kM}{B^2}(2XY)^{\frac{1}{2}}\right\}. \tag{39}$$

4. Leading-edge shock wave

The shape of a shock wave from the leading edge can be written in the form

$$F(x,y,t) = 0 = \frac{x}{B} - y - \bar{\xi}_s(y;\epsilon) - \delta\, \mathrm{e}^{\mathrm{i}kt}\tilde{\xi}_s(y;k,\epsilon) + \dots. \tag{40}$$

For an airfoil with upper surface defined by (28), with $a_1 > 0$, an analytical expression can be obtained for the first approximation to the steady-state shock-wave position $\xi = \bar{\xi}_s$. From (8) evaluated at the shock wave, together with the observation that in a first approximation a weak shock wave bisects the angle between the characteristics just upstream and just downstream, it is found that

$$\bar{\xi}_s = \frac{a_1}{2Ba_2}\left\{-1 + \left[1 - \frac{(\gamma+1)M^4 a_2 \epsilon y}{B^2}\right]^{\frac{1}{2}}\right\} + \dots. \tag{41}$$

This expression includes the solution for a wedge, since a straight line with the proper slope is recovered in the limit $a_2 \to 0$.

Next the largest term in $\tilde{\xi}_s$ can be found. The shock-wave speed is $c_n = -F_t/|\nabla F|$ and the normal component of the gas velocity just upstream is $q_n = F_x/|\nabla F|$. The difference $q_n - c_n$ is then the speed at which the shock wave moves into air at rest. For a weak shock wave the velocity change Δq_n imparted to the gas is $\Delta q_n = 4(q_n - c_n - 1/M)/(\gamma+1)$. Carrying out the necessary substitutions, and noting that $\tilde{\phi}_y \sim -B\tilde{\phi}_x$, one finds for the largest time-dependent terms

$$\tilde{\xi}_s' + \frac{\mathrm{i}kM^2}{B}\tilde{\xi}_s = \frac{(\gamma+1)M^4}{4B^3\xi_X} \mathrm{e}^{-\mathrm{i}(kM^2/B)(y+\bar{\xi}_s)} \tilde{\phi}_X, \tag{42}$$

where $\tilde{\phi}_X$ is evaluated with $x = B(y + \bar{\xi}_s)$. Integration then gives

$$\tilde{\xi}_s \mathrm{e}^{\mathrm{i}kM^2y/B} \sim -h(0) - \frac{(\gamma+1)M^4}{4B^4} V(0) \int_0^y \frac{1}{\xi_X} \mathrm{e}^{-\mathrm{i}k(\zeta + M^2\bar{\xi}_s/B)} J_0\left(\frac{kM}{B^2}\frac{(2XY)^{\frac{1}{2}}}{\epsilon^{\frac{1}{2}}}\right) \mathrm{d}y \tag{43}$$

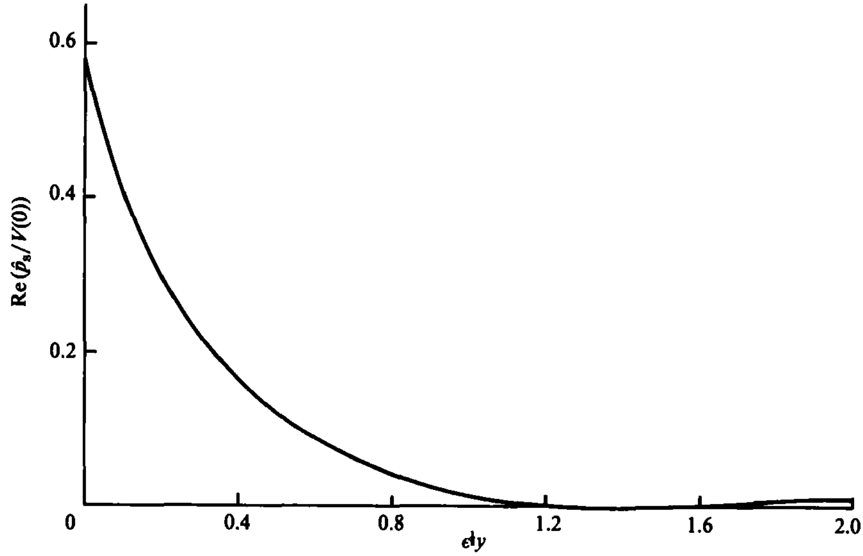

FIGURE 2. Decay of scaled unsteady pressure at leading-edge shock wave.

where X is evaluated at $x = B(y+\bar{\xi}_s)$. For consistency with the motion of the leading edge the integration constant has been chosen, with reference to (5) and (40), such that $\bar{\xi}_s = -h(0)$ at $y = 0$. The flow for $x, y = O(\delta)$ is easily shown to be quasi-steady, and no special discussion is needed for the first approximation close to the leading edge.

If $\epsilon^{\frac{1}{2}}y \rightarrow \infty$, provided that $k \gg \epsilon^{\frac{1}{2}}$, the integral in (43) approaches a constant of order $k^{-1}\epsilon^{-\frac{1}{2}}$, so that

$$\bar{\xi}_s \sim -\left\{\frac{M}{2k}\left(\frac{\gamma+1}{2Ba_1\epsilon}\right)^{\frac{1}{2}} V(0)+h(0)\right\} \mathrm{e}^{-\mathrm{i}kM^2y/B}+\dots \tag{44}$$

Thus at large distances, such that $y \gg \epsilon^{-\frac{1}{2}}$, a disturbance travels along the shock wave at a speed $\mathrm{d}y/\mathrm{d}t \sim B/M^2$. This is the value found for the simpler case of a cylindrical wave originating in a disturbance at the leading edge, with its centre convected downstream at the flow speed while its radius increases at the sound speed (e.g. Kurosaka 1977). The amplitude of the shock-wave oscillation for $y \gg \epsilon^{-\frac{1}{2}}$ approaches a constant value larger than that at the leading edge; while $|\bar{\xi}_s|$ grows indefinitely as y increases, $\delta|\bar{\xi}_s|$ increases from $O(\delta)$ to $O(\delta k^{-1}\epsilon^{-\frac{1}{2}})$. The instantaneous shock-wave velocity and slope, however, are related in such a way that the relative velocity $q_n - c_n$ (proportional to the right-hand side of (42)) associated with the terms shown in (44) is zero. The amplitude of the unsteady pressure jump should therefore be expected to decrease with increasing distance. From the jump conditions, the pressure p_s at the shock wave is found as $p_s = (q_n - c_n)\,\Delta q_n$. Substitution then gives the same leading term in the time-dependent pressure jump $\delta\,\mathrm{e}^{\mathrm{i}k(t-M^2x/B^2)}\,\tilde{p}_s$ as would be found from (39) with ξ replaced by $\bar{\xi}_s$ throughout. Thus $\tilde{p}_s$ remains $O(1)$ for $y = O(\epsilon^{-\frac{1}{2}})$ but is found to be $O(\xi_X^{-1}\,\epsilon^{-\frac{1}{4}}y^{-\frac{1}{2}})$ for $y \gg \epsilon^{-\frac{1}{2}}$. The real part of $\tilde{p}_s/V(0)$ as given by (39) at the shock wave is plotted against $\epsilon^{\frac{1}{2}}y$ in figure 2 for $\epsilon = 0.1$, $a_1 = 1$, $a_2 = -1$, $k = 1.0$ and $M = 2.0$. The factor $(B^2\xi_X)^{-1}$ decreases rapidly as y increases, and so the amplitude of the pressure oscillation is extremely small even at the first minimum of the factor J_0.

5. Concluding remarks

By a combined use of multiple scales and matched asymptotic expansions, a systematic description has been derived for the disturbances caused by an oscillating airfoil in supersonic flow, for reduced frequencies $k = O(1)$. In a numerical example, the oscillatory part of the pressure at a leading-edge shock wave is found to decrease fairly rapidly with increasing distance from the edge.

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